

Motivation

$X$  filtered topological space:

$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$$

-  $X$  is a "complicated" space, but each  $X^k$  is "just slightly more complicated" than  $X^{k-1}$ .

- more precisely: We know how to compute  $H_k(X^k, X^{k-1})$  for  $k \in \mathbb{N}$

- Question: Can we use this to (more easily) compute  $H_k(X)$ ?

Specific example

$X$  CW-complex, the filtration is given by the  $k$ -skeletons

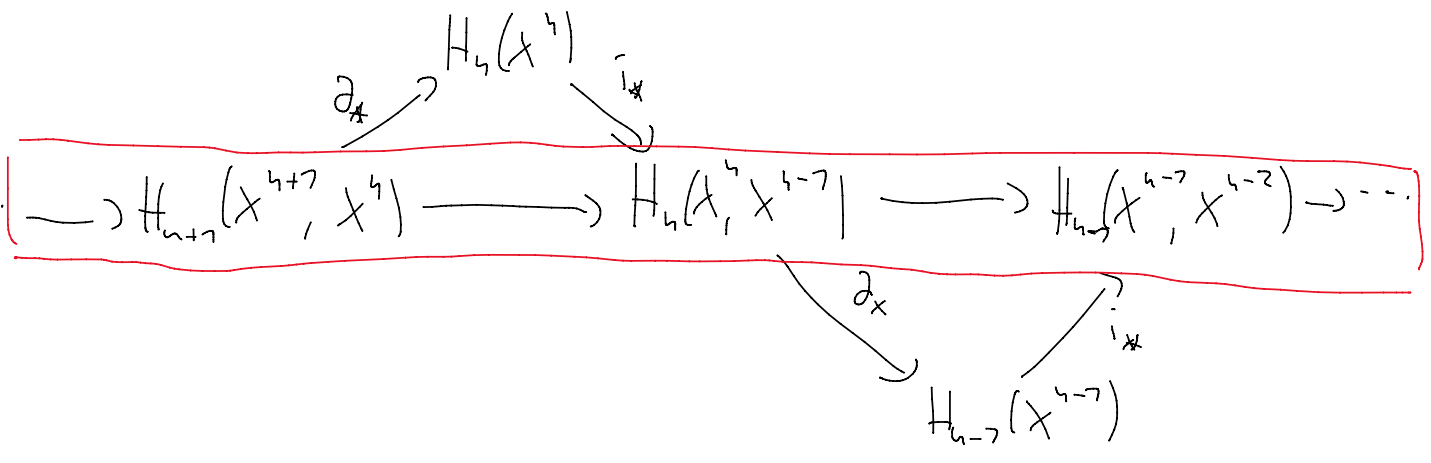
$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$$

To compute  $H_k(X)$ : Prove

**Lemma 2.34.** *If  $X$  is a CW complex, then:*

- (a)  $H_k(X^n, X^{n-1})$  is zero for  $k \neq n$  and is free abelian for  $k = n$ , with a basis in one-to-one correspondence with the  $n$ -cells of  $X$ .
- (b)  $H_k(X^n) = 0$  for  $k > n$ . In particular, if  $X$  is finite-dimensional then  $H_k(X) = 0$  for  $k > \dim X$ .
- (c) The map  $H_k(X^n) \rightarrow H_k(X)$  induced by the inclusion  $X^n \hookrightarrow X$  is an isomorphism for  $k < n$  and surjective for  $k = n$ .

(Hatcher)



## Definitions

- A differential group is an abelian group  $C$ , together with a homomorphism  $d: C \rightarrow C$  such that  $d^2 = d \circ d = 0$ .
- A graded group is an abelian group  $C$ , together with a collection of subgroups  $\{C_k \mid k \in \mathbb{Z}\}$  such that

$$C = \bigoplus_{k \in \mathbb{Z}} C_k$$

- A filtered group is an abelian group, together with a filtration  $0 = F_{-1}C \subseteq F_0C \subseteq F_1C \subseteq \dots \subseteq F_nC = C$  of subgroups.

Main example:

$X$  topological space:  $C_{\text{sing}}(X) := \bigoplus_{k \in \mathbb{Z}} \underbrace{C_{k, \text{sing}}(X)}_{k\text{-chain in } X}$  is a differential graded group.

If  $X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0$  is filtered, then  $C_{\text{sing}}(X)$  will become a filtered group by setting

$$F_p C_{\text{sing}}(X) := C_{\text{sing}}(X_p) = \bigoplus_{k \in \mathbb{Z}} C_{k, \text{sing}}(X_p)$$

Convention: Whenever an abelian group  $C$  has more than one of the structures defined above we require these to be compatible as follows:

- If  $C$  is differential graded group, we require the differential to be homogeneous:

$$\exists m \in \mathbb{Z} \text{ s.t. } \forall k \in \mathbb{Z}: d(C_k) \subseteq C_{k+m}$$

- If  $C$  is a differential filtered group, we require:

$$d(F_p C) \subseteq F_p C \quad \text{for all } p \in \mathbb{Z}.$$

- If  $C$  is a filtered graded group, we require

$$F_p C = \bigoplus_{k \in \mathbb{Z}} (F_p C \cap C_k)$$


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- For a differential group  $C$ , define

$$H(C) := \frac{\text{Ker } d}{\text{Im } d}$$

If  $C$  is graded, then  $H(C)$  inherits a grading:

$$H(C) = \bigoplus_{k \in \mathbb{Z}} H(C_k)$$

If  $C$  is filtered, then  $H(C)$  inherits a filtration:

$$F_p H(C) := \text{Image of } H(F_p C) \text{ under the induced morphism } H(F_p C) \rightarrow H(C).$$

- For a filtered group  $C$ , define

$$Gr(C) := \bigoplus_{p \in \mathbb{Z}} \frac{F_p C}{F_{p-1} C} \quad (\text{this is a graded group})$$

the associated graded group of  $C$

If  $C$  is a differential group, then  $Gr(C)$  will inherit a differential

$$d^0: Gr(C) \rightarrow Gr(C)$$

$$(d_p^0: \frac{F_p C}{F_{p-1} C} \rightarrow \frac{F_p C}{F_{p-1} C} \text{ for all } p \in \mathbb{Z})$$


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### The Spectral Sequence of a Filtered Differential Group

Throughout, let  $C$  be a filtered differential group.

We want to relate  $H(Gr(C))$  to  $H(C)$ .

Why aren't the two isomorphic?

First of all:  $H(Gr(C))$  is a graded group

$H(C)$  is a filtered group

Let's instead compare  $H(\mathcal{G}_r(C))$  to  $\mathcal{G}_r(H(C))$ .

Why aren't these groups isomorphic?

- The cycles in  $H_p(\mathcal{G}_r(C))$  are represented by elements of  $F_p C$  whose boundaries lie in  $F_{p-r} C$ .

The elements of  $\mathcal{G}_r(H(C))$  are represented by elements of  $H_p(C)$ . The cycles in  $H_p(C) = H(F_p C)$  are elements of  $F_p C$  whose boundary is  $\emptyset$ .

- The boundaries in  $H_p(\mathcal{G}_r(C))$  are boundaries of elements in  $F_p C$ .

The boundaries in  $(\mathcal{G}_r(H(C)))_p$  are boundaries of elements of  $C$ .

↳ so in general, these groups are not isomorphic.

(Informal) idea/claim: These two problems are all you have to fix.

Idea: Find a gradual transition from  $H(\mathcal{G}_r(C))$  to  $\mathcal{G}_r(H(C))$  by gradually decreasing the number of cycles, and increasing the number of boundaries.

Rewrite the definition for  $H_p(\mathcal{G}(C))$ :

$$H_p(\mathcal{G}_r(C)) = \frac{F_p C \cap d^{-r}(F_{p-r} C)}{[F_{p-r} C \cap d^{-r}(F_{p-r} C)] + [F_p C \cap d(F_p C)]}$$

Definition: (let  $C$  be a filtered differential group),

For  $r \in \mathbb{Z}$ , define

$$E_p^{-r} := \frac{F_p C \cap d^{-r}(F_{p-r} C)}{[F_{p-r} C \cap d^{-r}(F_{p-r} C)] + [F_p C \cap d(F_{p+r-r} C)]} \quad \forall p \in \mathbb{Z}$$





Cor from  $\{C\}$ .

Back to our original motivation:

$X$  CW-complex,  $\emptyset = X^{-1} \subseteq X^0 \subseteq \dots \subseteq X^p = X$  skeletons

$$E_{pq}^0 = C_q^{sing}(X^p) / C_{q-1}^{sing}(X^p) \underbrace{C_p(X)}_{\{p\text{-cells}\}}$$

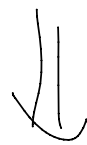
$$E_{pq}^1 = H_{p+q}(X^p, X^{p-1}) = \begin{cases} \mathbb{Z} & \text{if } q=0 \\ 0 & \text{otherwise} \end{cases}$$

Draw a table:

	$q$	$0$	$0$	$0$	$0$
$1$		$0 \leftarrow$	$0 \leftarrow$	$0 \leftarrow$	$0$
$0$		$C_0(X) \leftarrow$	$C_1(X) \leftarrow$	$C_2(X) \leftarrow$	$C_3(X)$
		$0$	$1$	$2$	$3$

(zeros)

$E^1$



	$q$	$0$	$0$	$0$	$0$
$1$		$0 \leftarrow$	$0 \leftarrow$	$0 \leftarrow$	$0$
$0$		$H_0^{cell}(X) \leftarrow$	$H_1^{cell}(X) \leftarrow$	$H_2^{cell}(X) \leftarrow$	$H_3^{cell}(X)$
		$0$	$1$	$2$	$3$

zeros

$E^2$

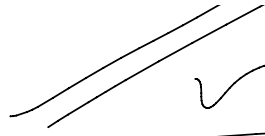
no more nontrivial homology, so the sequence stabilizes here.

$$\Rightarrow \frac{H_n(X^n)}{\bigoplus H_n(X^{n-1})} = E_{n0}^\infty = E_{n0}^2 = H_n^{cell}(X)$$

$\vdots$



$$H_n(X)$$



Question

$$C = \bigoplus_{k \in \mathbb{Z}} C_k \quad \text{graded group}$$

$$\dots \subseteq \bigoplus_{k=-\infty}^{\infty} C_k \subseteq \bigoplus_{k=-\infty}^{\infty} C_k \subseteq \bigoplus_{k=-\infty}^{\infty} C_k \subseteq \dots$$

