Spectral Sequences 3

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Motivation

X filtered topological space:

$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X$$

- X is a "complicated" space, but each X^k is "just slightly more complicated" than X^{k-1},
- more precisely: We know how to compute $H_\bullet(X^k, X^{k-1})$ for $k \in \mathbb{N}$
- Question: Can we use this to (more easily) compute $H_\bullet(X)$?

Specific example

X CW-complex, the filtration is given by the k-skeletons

$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots$$

To compute $H_{sing}(X)$: Prove

Lemma 2.34. *If X is a CW complex, then:*

(a) *$H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.*

(b) *$H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.*

(c) *The map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.*

(Hatcher)

$$\begin{tikzcd}
 & & H_n(X^n) \arrow[dr, "i_*"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_*"] \arrow[rr] & & H_n(X^n, X^{n-1}) \arrow[rr] \arrow[dr, "\partial_*"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "i_*"'] & &
\end{tikzcd}$$

Definitions

- A *differential group* is an abelian group C, together with a homomorphism $d: C \to C$ such that $d^2 = d \circ d = 0$.

- A *graded group* is an abelian group C, together with a collection of subgroups $\{C_k \mid k \in \mathbb{Z}\}$ such that

$$C = \bigoplus_{k \in \mathbb{Z}} C_k$$

- A *filtered group* is an abelian group, together with a filtration $0 = F_{-1}C \subseteq F_0 C \subseteq F_1 C \subseteq \cdots \subseteq F_n C = C$ of subgroups.

Main example:

X topological space: $C_{sing}(X) := \bigoplus_{k \in \mathbb{Z}} \underbrace{C_{k,sing}(X)}_{k\text{-chains in } X}$ is a differential graded group.

If $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$ is filtered, then $C_{sing}(X)$ will become a filtered group by setting

$$F_p C_{sing}(X) := C_{sing}(X_p) = \bigoplus_{k \in \mathbb{Z}} C_{k,sing}(X_p)$$

Convention: Whenever an abelian group C has more than one of the structures defined above, we require these to be compatible as follows:

- If C is differential graded group, we require the differential to be homogeneous:

$$\exists\, m \in \mathbb{Z} \text{ s.t. } \forall k \in \mathbb{Z}: \quad d(C_k) \subseteq C_{k+m}$$

- If C is a differential filtered group, we require:

$$d(F_p C) \subseteq F_p C \qquad \text{for all } p \in \mathbb{Z}.$$

- If C is a filtered graded group, we require
$$F_p C = \bigoplus_{k \in \mathbb{Z}} (F_p C \cap C_k)$$

- For a differential group C, define
$$H(C) := \frac{\operatorname{Ker} d}{\operatorname{Im} d}$$
If C is graded, then $H(C)$ inherits a grading:
$$H(C) = \bigoplus_{k \in \mathbb{Z}} H(C_k)$$
If C is filtered, then $H(C)$ inherits a filtration:
$$F_p H(C) := \text{Image of } H(F_p C) \text{ under the induced morphism } H(F_p C) \to H(C).$$

- For a filtered group C, define
$$Gr(C) := \bigoplus_{p \in \mathbb{Z}} \frac{F_p C}{F_{p-1} C} \qquad \text{(this is a graded group)}$$
the associated graded group of C

If C is a differential group, then $Gr(C)$ will inherit a differential $d^0 : Gr(C) \to Gr(C)$
$$\left(d^0_p : \frac{F_p C}{F_{p-1} C} \to \frac{F_p C}{F_{p-1} C} \text{ for all } p \in \mathbb{Z} \right)$$

The Spectral Sequence of a Filtered Differential Group

Throughout, let C be a filtered differential group.

We we want to relate $H(Gr(C))$ to $H(C)$.

Why aren't the two isomorphic?

First of all: $H(Gr(C))$ is a graded group

$H(C)$ is a filtered group

Let's instead compare $H(Gr(C))$ to $Gr(H(C))$.
Why aren't these groups isomorphic?

- The cycles in $H_p(Gr(C))$ are represented by elements of F_pC whose boundaries lie in $F_{p-1}C$.

  The elements of $Gr(H(C))$ are represented by elements of $H_p(C)$. The cycles in $H_p(C) = H(F_pC)$ are elements of F_pC whose boundary is 0.

- The boundaries in $H_p(Gr(C))$ are boundaries of elements in F_pC.

  The boundaries in $(Gr(H(C)))_p$ are boundaries of elements of C

↳ so in general, these groups are not isomorphic.

(Informal) idea/claim: These two problems are all you have to fix.

<u>Idea</u>: Find a <u>gradual transition</u> from $H(Gr(C))$ to $Gr(H(C))$ by gradually decreasing the number of cycles, and increasing the number of boundaries.

Rewrite the definition for $H_p(Gr(C))$:

$$H_p(Gr(C)) = \frac{F_pC \cap d^{-1}(F_{\boxed{p-1}}C)}{[F_{p-1}C \cap d^{-1}(F_{\boxed{p-1}}C)] + [F_pC \cap d(F_{\boxed{p}}C)]}$$

<u>Definition</u>: (Let C be a filtered differential group).

For $r \in \mathbb{Z}$, define

$$E_p^r := \frac{F_pC \cap d^{-1}(F_{\boxed{p-r}}C)}{[F_{p-1}C \cap d^{-1}(F_{\boxed{p-r}}C)] + [F_pC \cap d(F_{\boxed{p+r-1}}C)]} \quad \forall p \in \mathbb{Z}.$$

$$[F_{p-1}C \cap d(F_{\boxed{p-r}}C)] + [F_pC \cap d(F_{\boxed{p+r-1}}C)]$$

$$\underbrace{E^r}_{} := \bigoplus_{p\in\mathbb{Z}} E^r_p$$

↳ "the r-th page of the spectral sequence of C".

If C is also graded, then E^r_p is also graded:

$$E^r_{pq} := \frac{F_pC_{p+q} \cap d^{-1}(F_{\boxed{p-r}}C_{p+q+1})}{[F_{p-1}C \cap d^{-1}(F_{\boxed{p-r}}C_{p+q+1})] + [F_pC \cap d(F_{\boxed{p+r-1}}C_{p+q-1})]} \quad \forall\, p\in\mathbb{Z}.$$

then $E^r_p = \bigoplus_{q\in\mathbb{Z}} E^r_{pq}$

Observe

$$E^1_p = H_p(Gr(C)) \quad \Rightarrow \quad E^1 = H(Gr(C)).$$

furthermore

$$E^0_p = \frac{F_pC}{F_{p-1}C} \quad \Rightarrow \quad E^0 = Gr(C)$$

and: for $r \gg 0$: $F_{p-r}C = \underline{0}$, $F_{p+r-1}C = C$

(if the filtration is finite).

so

$$E^r_p = E^{r+1}_p = \dots = \frac{F_pC \cap d^{-1}(0)}{[F_{p-1}C \cap d^{-1}(0)] + [F_pC \cap d(C)]}$$

$$= \frac{H(F_pC)}{H(F_{p-1}C)} =: E^\infty_p,$$

so $E^\infty := \bigoplus_{p\in\mathbb{Z}} E^\infty_p = Gr(H(C))$.

"the spectral sequence stabilizes"

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d induces a differential on E^r ...

d induces a differential $d_p^r : E_p^r \to E_{p-r}^r$
(and hence $d^r : E^r \to E^r$ homogeneous of degree $-r$):
Take $\alpha \in E_p^r$. Then α is represented by

$$a \in F_p C \cap d^{-1}(F_{p-r} C). \quad \text{Then}$$

$d(a) \in F_{p-r} C$, and $d(d(a)) = d^2(a) = 0 \in F_{p-2r} C$, so

$$d(a) \in \underbrace{F_{p-r} C \cap d^{-1}(F_{(p-r)-r} C)}_{\text{numerator of } E_{p-r}^r}.$$

Define $d_p^r(\alpha)$ to be the equivalence class of $d(a)$ in E_{p-r}^r.

<u>Main theorem</u>

1) d_p^r is well-defined.

2) $d_{p-r}^r \circ d_p^r = 0$ (so $d^r \circ d^r = 0$, so E^r is canonically a differential group)

3) $\dfrac{\operatorname{Ker} d_p^r}{\operatorname{Im} d_{p+r}^r} \cong E_p^{r+1}$ (in part. $H(E^r) \cong E^{r+1}$). canonically

<u>Proof</u> diagram chase

<u>Result</u> Knowing E^1 (or some other E^r) as well as d, we can compute E^2, then E^3, ..., until the sequence stabilizes $\leadsto$ obtain E^∞

$\leadsto$ we can compute $Gr(H(C))$ from $H(Gr(C))$ (or from $Gr(C)$).

Cor from Sr(C)).

Back to our original motivation:

X CW-complex, $\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^n = X$ skeletons

$$E^0_{pq} = C^{sing}_q(X^p) / C^{sing}_{q-1}(X^{p-1})$$

$$E^1_{pq} = H_{p+q}(X^p, X^{p-1}) = \begin{cases} \overbrace{\mathbb{Z}^{\{p\text{-cells}\}}}^{C_p(X)}, & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a table:

$$E^1: \quad \begin{array}{c|cccc}
q & 0 & 0 & 0 & 0 \\
1 & 0 \leftarrow & 0 \leftarrow & 0 \leftarrow & 0 \\
0 & C_0(X) \leftarrow & C_1(X) \leftarrow & C_2(X) \leftarrow & C_3(X) \\
\hline
 & 0 & 1 & 2 & 3 \; p
\end{array}$$

(zeroes)

$$\Downarrow$$

$$E^2: \quad \begin{array}{c|cccc}
q & 0 & 0 & 0 & 0 \\
 & 0 & 0 & & \\
 & H_0^{cell}(X) & H_1^{cell}(X) & H_2^{cell}(X) & H_3^{cell}(X) \\
\hline
 & 0 & 1 & 2 & 3 \; p
\end{array}$$

zeroes

no more nontrivial homology, so the sequence stabilizes here.

$$\Rightarrow \frac{H_n(X^n)}{\underbrace{H_n(X^{n-1})}_{0}} = E^\infty_{n0} = E^2_{n0} = H_n^{cell}(X)$$

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$H_4(X^4)$

Question

$$C = \bigoplus_{k \in \mathbb{Z}} C_k \qquad \text{graded group}$$

$$\cdots \subseteq \bigoplus_{k=-\infty}^{1} C_k \subseteq \bigoplus_{k=-\infty}^{2} C_k \subseteq \bigoplus_{k=-\infty}^{3} C_k \subseteq \cdots$$